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Escape of Brownian particles in an asymmetric bistable sawtooth potential driven by cross-correlated noises

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Abstract. The relative escape rate (RER) for Brownian particles in an asymmetric bistable sawtooth potential driven by cross correlations between multiplicative white noise and additive white noise is studied. A new expression of the mean first-passage time is derived under the condition of piecewise linear potentials and discontinuous diffusion function. Based on the results of RER numerically calculated, it is found that (i) under positively correlated noises action (*i.e.* $\lambda > 0$, and λ is the correlation strength), the escape rate exhibits the suppression platform as the intensity of multiplicative noise varies. The effect of suppression becomes more pronounced with the growth of height of the deterministic potential barrier for transition, and with the increase of λ. However, for the case of uncorrelated noises ($\lambda = 0$) and of negatively correlated noises ($\lambda < 0$), the suppression platform disappears. (ii) The positive correlation between noises amplifies the change of the escape rate with the height of barrier for transition, while the negative correlation between noises suppresses this change.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

The study of the escape problem of Brownian particles has attracted a great deal of interests for many years, because the escape time is of physical importance to characterize the dynamics of a system driven by noises. Resonant activation, giant suppression and some novel phenomena were found [1–6]. Since Fulinski and Telejko in 1991 proposed the idea of cross-correlation between additive and multiplicative noises [7], the effects of the cross-correlated noises on the dynamics of systems have been paid much attention, and this ideal of the cross-correlated noises has been generalized to the other subjects of stochastic systems such as stochastic resonance [8–10].

Recently, Jia *et al.* [11] studied the effects of correlated additive and multiplicative noises on the mean first-passage time (MFPT) of the symmetric bistable potential model,

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4},$$

for the case of zero correlation time. It was found that the MFPT is affected by the correlation strength λ. In the presence of perfectly correlated noises ($\lambda = 1$), MFPT corresponding to $\alpha > D$ and $\alpha < D$ (α and D stands for the strengths of additive and multiplicative noises respectively) exhibits very different behaviors, and the MFPT for $\alpha = D$ diverges to infinity [11]. Mei *et al.* [12] examined the MFPT of the same model for the case of nonzero correlation time. It was showed that τ (the correlation time) and λ play opposing roles in the MFPT and the MFPT as a function of τ is nonmonotonic [12]. Madureira *et al.* [2] proved that in the same potential, as a function of the correlation strength between additive and multiplicative noise sources the activate rate (defined as the inverse of the MFPT) can be suppressed by order of magnitude [2]. The potential application of a bistable model subject to correlated noises is given by the switching of magnetization in single-domain ferromagnetic particles, in which external and internal magnetic field fluctuations are generally correlated and mutually influence the bistable relaxation dynamics of the magnetic moment [2]. However, previous works were only concerned with the symmetric bistable potential model. A natural question turns up, that is in an asymmetric bistable potential the effects of correlation between noises on the escape for Brownian particles.

In the present paper, the escape rate of Brownian particles in an asymmetric sawtooth potential coupled to correlated multiplicative and additive white noises is investigated. Our main aim is, by this exact soluble model, to exhibit the impact of correlated noises on the escape rate for the case of an asymmetric bistable potential. In Section 2, a new expression of the mean first-passage time is derived under the condition of piecewise linear potentials and an exact analytic expressions of the relative rate is

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derived. A discussion of the results concludes the paper in Section 3.

2 MFPT of an asymmetric bistable sawtooth potential

We consider an exact soluble model, *i.e.* an asymmetric bistable sawtooth system with with two cross-correlated zero-mean Gaussian noises. The corresponding Langevin equation reads

$$\dot{x} = -\varphi'(x) + u(x)p(t) + q(t), \tag{1}$$

here $p(t)$ and $q(t)$ are Gaussian white noises with

$$\langle q(t)\rangle = \langle p(t)\rangle = 0, \tag{2a}$$
$$\langle q(t)q(t')\rangle = 2D\delta(t-t'), \tag{2b}$$
$$\langle p(t)p(t')\rangle = 2Q\delta(t-t'), \tag{2c}$$

and

$$\langle q(t)p(t')\rangle = \langle p(t)q(t')\rangle = 2\lambda\sqrt{DQ}\delta(t-t'), \tag{2d}$$

in which Q stands for the strength of the multiplicative noise $p(t)$. D is the strength of the additive noise $q(t)$ and λ measures the degree of correlation between $p(t)$ and $q(t)$. The potential function $\varphi(x)$ is given by

$$\varphi(x) = \begin{cases} -\dfrac{2b}{L}x - 2b & -\infty < x \le -\dfrac{L}{2} + \varepsilon \\ \dfrac{2b}{L}x - \dfrac{4b}{L}\varepsilon & -\dfrac{L}{2} + \varepsilon < x \le 2\varepsilon \\ -\dfrac{2b}{L}x + \dfrac{4b}{L}\varepsilon & 2\varepsilon < x \le \dfrac{L}{2} + \varepsilon \\ \dfrac{2b}{L}x - 2b & \dfrac{L}{2} + \varepsilon < x < \infty. \end{cases} \tag{3}$$

From the deterministic point of view, the stable steady states correspond to minima of the potential function, and the unstable ones correspond to maxima of the potential function. One finds two stable steady states, located at $x_1 = -L/2 + \varepsilon$, $x_2 = L/2 + \varepsilon$ respectively. The unstable state is located at $x_0 = 2\epsilon$. Obviously, when $\varepsilon = 0$, $\varphi(x)$ is just a symmetric bistable potential. When $\epsilon \neq 0$, $\varphi_1 = \varphi(x_1) = -b - 2b\varepsilon/L$. $\varphi_2 = \varphi(x_2) = -b + 2b\varepsilon/L$. The potential difference of the two well $\Delta\varphi = -4b\varepsilon/L$ exists and $\varphi(x)$ is an asymmetric bistable potential. Increasing ϵ means that the height of the barrier for transition from x_1 to x_2, $-\varphi_1$, grows high.

The Fokker-planck equation associated with equations (1, 2) can be written as Fox's equation for the probability distribution [13] in the limit of correlation times going to zero

$$\partial_t W(x,t) = -\partial_x A(x)W(x,t) + \partial_x^2 h^2(x)W(x,t), \tag{4}$$

with

$$A(x) = -\varphi'(x) + h(x)h'(x), \tag{5}$$

and

$$h(x) = \left[Qu^2(x) + 2\lambda\sqrt{QD}u(x) + D\right]^{1/2}. \tag{6}$$

Equations (4–6) can also readily obtained using the method in reference [14].

In essence, from the forward FPF, equation (4),the backward FPE can be obtained, and then the MFPT equation of T is derived [15]

$$A(x)\partial_x T(x) + h^2(x)\partial_x^2 T(x) = -1. \tag{7}$$

For the sake of mathematical simplicity, assume the coefficient of the multiplicative noise has the following form [16]

$$u(x) = \begin{cases} c, & -\infty < x \le -\dfrac{L}{2} + \varepsilon, 2\varepsilon < x \le \dfrac{L}{2} + \varepsilon \\ -c, & -\dfrac{L}{2} + \varepsilon < x \le 2\varepsilon, \dfrac{L}{2} + \varepsilon < x < \infty. \end{cases} \tag{8}$$

We note the fact that the potential function described by (3) is piecewise smooth and that the diffusion coefficient $h^2(x)$ have a finite jump at piecewise points of the function (3). Assume that $T(x)$ should be continuous and smooth, and under the reflective boundary at $x = -\infty$ and an absorbable boundary at $x = x_2$, by solving equation (7) we get the expression of the MFPT from x ($x \le x_1$) to x_2

$$\begin{aligned} T(x) &= \int_x^{x_1} \frac{\mathrm{d}y}{\psi(y)} \int_{-\infty}^{y} \frac{\psi(z)}{h^2(z)}\mathrm{d}z \\ &+ \int_{x_1}^{x_0} \frac{\mathrm{d}y}{\psi(y)} \left\{ \int_{x_1}^{y} \frac{\psi(z)}{h^2(z)}\mathrm{d}z + \frac{\psi(x_1^+)}{\psi(x_1^-)} \int_{-\infty}^{x_1} \frac{\psi(z)}{h^2(z)}\mathrm{d}z \right\} \\ &+ \int_{x_0}^{x_2} \frac{\mathrm{d}y}{\psi(y)} \left\{ \int_{x_0}^{y} \frac{\psi(z)}{h^2(z)}\mathrm{d}z + \frac{\psi(x_0^+)}{\psi(x_0^-)} \int_{x_1}^{x_0} \frac{\psi(z)}{h^2(z)}\mathrm{d}z \right. \\ &+ \left. \frac{\psi(x_0^+)}{\psi(x_0^-)} \frac{\psi(x_1^+)}{\psi(x_1^-)} \int_{-\infty}^{x_1} \frac{\psi(z)}{h^2(z)}\mathrm{d}z \right\}, \end{aligned} \tag{9}$$

where

$$\psi(x) = \int^x \frac{A(y)}{h^2(y)}\mathrm{d}y. \tag{10}$$

In equation (9), $\varphi(x^\pm)$ is abbreviated from $\lim_{\tau\to 0} \varphi(x \pm |\tau|)$. Note that if $\varphi(x)$ and $h(x)$ are continuous and smooth, equation (9) reduces to the formula of MFPT given by reference [17].

Consider now Brownian motion from the minimum of the first well at x_1 to the minimum of the second well at x_2. Using equation (9) with equation (8), we finally obtain the MFPT

$$\begin{aligned} T(\varepsilon) &= 2\gamma D\mu_1 \left[\exp\left(-\frac{\varphi_1\mu_2^{-1}}{D}\right) - 1\right] \\ &\times \left[1 - \exp\left(\frac{\varphi_2\mu_1^{-1}}{D}\right)\right] \\ &+ 2\gamma D\mu_2 \left[\exp\left(-\frac{\varphi_1\mu_2^{-1}}{D}\right) - 1\right] + \gamma\Delta\varphi, \end{aligned} \tag{11}$$

where $\gamma = L^2/4b^2$. μ_i are given by

$$\mu_i = \frac{Qc^2 \pm 2\lambda c\sqrt{QD} + D}{D}, \tag{12}$$

in which $i = 1$ corresponds to $+$ and $i = 2$ to $-$. When the parameter ε=0, φ_1=φ_2=φ_0 (symmetric potential), the MFPT reduces to

$$\begin{aligned} T(0) = 2\gamma D\mu_1 &\left[\exp\left(-\frac{\varphi_0\mu_2^{-1}}{D}\right) - 1\right] \\ &\times \left[1 - \exp\left(\frac{\varphi_0\mu_1^{-1}}{D}\right)\right] \\ &+ 2\gamma D\mu_2 \left[\exp\left(-\frac{\varphi_0\mu_2^{-1}}{D}\right) - 1\right]. \end{aligned} \tag{13a}$$

For $\lambda = 0$, in the absence of correlation between additive and multiplicative noises, $\mu_1 = \mu_2 = \mu$ and equation (11) reduces to

$$\begin{aligned} T_{\lambda=0} = 2\gamma D\mu &\left[\exp\left(-\frac{\varphi_1\mu^{-1}}{D}\right) - 1\right] \\ &\times \left[2 - \exp\left(\frac{\varphi_2\mu^{-1}}{D}\right)\right] + \gamma\Delta\varphi. \end{aligned} \tag{13b}$$

When there only exists the additive noise, *i.e.*, $Q = 0$, inserting $\mu_1 = \mu_2 = 1$ in equation (11) we have

$$\begin{aligned} T_{Q=0} = 2D\gamma &\left[\exp\left(-\frac{\varphi_1}{D}\right) - 1\right]\left[2 - \exp\left(\frac{\varphi_2}{D}\right)\right] \\ &+ \gamma\Delta\varphi. \end{aligned} \tag{13c}$$

Furthermore, we can get the relative escape rate

$$\nu_1 = \frac{T_{Q=0}}{T(\varepsilon)}, \tag{14a}$$

and the relative escape ν_2 which is defined as

$$\nu_2 = \frac{T(0)}{T(\varepsilon)} \tag{14b}$$

which may reflects the influence of the height alteration of the barrier over which particles get on the escape rate.

3 Discussions and conclusions

To illustrate the effects of correlated noises on the escape rate in an asymmetric bistable potential, the relative escape rates, ν_1 and ν_2, are numerically calculated for different parameters.

As can be seen clearly from the Figure 1, when the correlation between noises is positive ($\lambda > 0$), the relative escape rates ν_1 in an asymmetric ($e.g.\ \epsilon = 0.4, or -0.4$) bistable potential exhibits one minimum value, namely suppression effect. Increasing λ intensifies the suppression of escape rate. Also we see that the higher the deterministic barrier that particles cross is, the more intense the suppression is (*i.e.* there is a more pronounced minimum value for large ϵ.). However, the suppression phenomenon disappears whether under the uncorrelated noises action($\lambda = 0$, see Fig. 1), or the negatively correlated noises action ($\lambda < 0$, see Fig. 2). From the physical side, $-\varphi_1$ is the height of the deterministic barrier for transition from x_1 to x_2, and $-\varphi_1\mu_2^{-1}/D$ is the effective barrier in the existence of noises. Thus the behaviors of the escape rate mainly depends on the activation factor $\exp(\varphi_1\mu_2^{-1}/D)$, weakly depends on the depth of the second well. We note that μ_2 has a minimum (μ_{2m}) which is $1 - \lambda^2$ at $Q_c = \lambda^2 D/c^2$ when $\lambda > 0$. In particular, when $\lambda = 1$ and $Q = D/c^2$, μ_{2m} equals to zero, so the factor $\exp(\varphi_1\mu_2^{-1}/D)$ approaches zero (since $\varphi_1 < 0$), and then the escape for particles cannot take place. This coincides with the result ($T = \infty$) obtained from equation (11). Hence, if Q varies near Q_c, the escape rate is rather small, and so the suppression platform comes about. When increasing λ and $\lambda > 0$, the height of the effective barrier $-\varphi_1\mu_{2m}^{-1}/D$ rapidly increases, the escape rate is dramatically suppressed. Similarly, the escape rate may drop to the lower point for larger ϵ because the factor $\exp(\varphi_1\mu_{2m}^{-1}/D)$ with large ϵ is smaller than that with small ϵ.

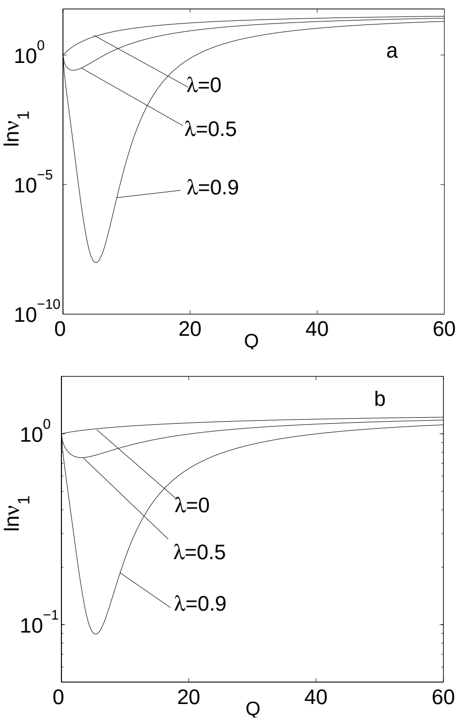

Fig. 1. Relative escape rate ν_1 as a function of Q for the case of $\lambda \geq 0$. $b = 0.25$, $c = 0.125$, $D = 0.1$, $L = 1$. (a) $\epsilon = 0.4$. (b) $\epsilon = -0.4$.

For fixed ϵ and increasing Q, the influence of λ on μ_1 and μ_2 becomes weak and weak, and thus the ν_1 tends to the sameness, which is illustrated in Figures 1 and 2.

Not unexpectedly, the escape rate ν_2 as a function of ϵ decays as ϵ increases no matter what λ is, as can be seen from Figure 3. This implies that particles always take much more time to pass through a higher potential barrier.

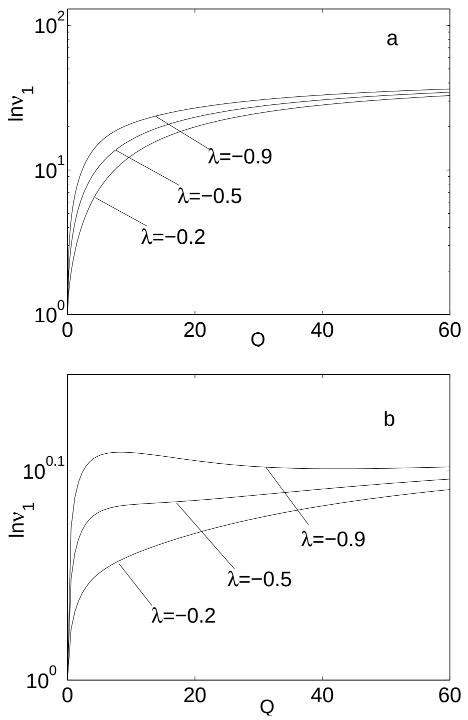

Fig. 2. Relative escape rate ν_1 as a function of Q for the case of $\lambda < 0$. $b = 0.25$, $c = 0.125$, $D = 0.1$, $L = 1$. (a) $\epsilon = 0.4$. (b) $\epsilon = -0.4$.

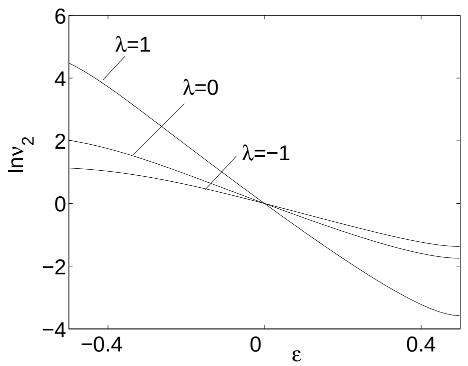

Fig. 3. Relative escape rate ν_2 as a function of ϵ for the different values of λ. $Q = 0.5$. $b = 0.25$, $c = 0.125$, $D = 0.1$, $L = 1$.

Figure 3 also shows that under the positively correlated noises action, λ enlarges the effect caused by changing the height of barrier over which particles surmount on the escape rate, but λ suppresses the effect in the case with negatively correlated noises ($\lambda < 0$), which can be more clearly seen in the Figure 4.

Finally, we remark that for a general bistable potential subject to correlated noises, the MFPT from one deterministic steady-state to the other mainly depends on the height of barrier for the transition in the effective potential associated with force [2]

$$\frac{D\varphi(y)}{h^2(y)} = \frac{D\varphi(x)}{Qu^2(x) + 2\lambda\sqrt{QD}u(x) + D}. \tag{15}$$

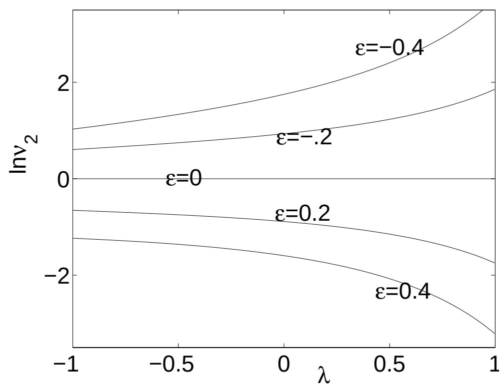

Fig. 4. Relative escape rate ν as a function of λ for the different values of ϵ. $Q = 0.5$, $b = 0.25$, $c = 0.125$, $D = 0.1$ and $L = 1$.

Obviously, if the positive correlated noises make the denominator of (15) decrease, the negative correlated noises make it increase. Hence, $\lambda > 0$ and $\lambda < 0$ play opposing roles on the effect caused by the height alteration of the barrier for transition on the escape rate, even though the coefficient of multiplicative noise $u(x)$ is continuous.

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